

Title: Iterating the Function, $f(x) = x^2 + c$, Over the Real Numbers

Brief Overview:

Students will use a spreadsheet to study the sequences of real numbers obtained by iterating the function $f(x) = x^2 + c$ for various values of c and initial point x_0 . They will discover the various types of behavior that can result and when it occurs. They will use algebra to prove their conjectures.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*
- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*
- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*
- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*
- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*
- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

• Content Standards

Algebra

Students will investigate functions from the perspective of iteration. They will study sequences generated by iteration, and find their limits if they exist. They also will solve equations and work with algebraic expressions to prove their conjectures.

• Process Standards

Problem Solving

Students will solve mathematical problems by using spreadsheets to search for patterns, make conjectures, and prove them algebraically.

Reasoning and Proof

Students will use inductive reasoning to discover how the value of c affects the behavior of the function under iteration. Then they will use deductive reasoning to prove some of their findings algebraically.

Communication

Students will explain relevant terms and concepts, as well as their findings, both orally and in writing.

Connections

Students will see connections between algebra, discrete mathematics, and analysis.

Representation

Students will use time-series graphs to picture the behavior of functions under iteration, and develop an intuitive understanding of attracting and repelling fixed points, cycles, chaos, and basins of attraction.

Links to Maryland High School Mathematics Core Learning Goals**Functions and Algebra**

- **1.1.2**

Students will iterate functions, solve quadratic equations, and find the expression for the fixed points of quadratic functions in terms of their constant coefficient.

Geometry, Measurement, and Reasoning

- **2.2.3**

Students will use inductive reasoning to discover how the value of c affects the behavior of the function under iteration. Then they will use deductive reasoning to prove some of their findings algebraically.

Grade/Level:

Grades 9-12

Duration/Length:

This activity will take four or five days, depending on class duration and students' background knowledge.

Prerequisite Knowledge:

Students should have working knowledge of the following concepts:

- Functions and function notation
- Graphing on the coordinate plane
- Quadratic equations and the quadratic formula

Student Outcomes:

Students will:

- use a spreadsheet to investigate the orbits of the function $f(x) = x^2 + c$ under iteration.
- make time-series graphs to represent the orbits and determine their behavior.
- use inductive reasoning to make conjectures about the resulting behavior.
- use algebra to prove some of those conjectures.

Materials/Resources/Printed Materials:

- Calculators
- Graph paper
- Computers with spreadsheet software

Development/Procedures:

Prepare a spreadsheet file ahead of time as follows:

	A	B
1	c =	
2		
3	Iteration no.	Orbit
4	0	0
5	=a4+1	=b4*b4+b\$1

Copy row 5 to rows 6 through 104. This will calculate the first 100 iterations of the function $f(x) = x^2 + c$, where the value of c is entered by the user in cell B1, and the initial value x_0 is entered in cell B4. Create a time-series graph of these iterates by setting the x-series to cells A4 through A104 and the y-series to cells B4 through B104. The specific commands to do this vary depending on the software used, so check your documentation if necessary. Position the graph so it is visible to the right of column B. Make this spreadsheet available on each computer your students will use.

Begin class by defining iteration: Given a function f and an **initial point** x_0 , form a sequence as follows: Let $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_{n+1} = f(x_n)$. This process is called **iteration**, and the resulting sequence is called the **orbit** of x_0 .

As an example, consider the function $f(x) = x^2$. Have students work in pairs to find the first five terms of the orbit, assigning a different initial point greater than 1 to each pair.

They should notice that the terms increase without bound. Ask them if this is always true. Have them try values for x_0 that are between 0 and 1.

They should notice that the orbits now converge to 0. Ask them why this occurs. Ask them what happens when $x_0 = 0$ and $x_0 = 1$. Then have them explore negative values of x_0 , and explain their results.

Summarize the discussion by defining the following terms:

- A **fixed point** for the function f is a number that solves the equation $f(x) = x$. 0 and 1 are fixed points for $f(x) = x^2$ since $f(0) = 0$ and $f(1) = 1$. In other words, solving the equation $x^2 = x$ results in the two solutions $x = 0$ or $x = 1$.
- An **attracting fixed point**, or **attractor**, is a fixed point to which all orbits beginning from an initial point close enough converge to it. In this example, 0 is an attractor since all orbits starting between -1 and 1 converge to 0.
- A **repelling fixed point** is a fixed point such that orbits with nearby initial points diverge from it. 1 is a repelling fixed point, since orbits starting near it move away from it and toward either infinity or the attractor 0.
- The **basin of attraction** is the set of all initial points whose orbits converge to a particular attractor. For this function, the open interval $-1 < x < 1$ is the basin of attraction for the attractor 0. Discuss why this is the basin of attraction. [Multiplying by a number less than 1 in absolute value results in a product whose magnitude is smaller than that of the original number. Thus, repeatedly squaring a number less than one in absolute value results in terms of decreasing magnitude.]

Show students how to make a **time-series graph** by plotting the values x_n of the orbit against the iteration numbers n . Have them plot the orbits with $x_0 = 0, .5, .9, 1, 1.1$, and 1.5 , for $n = 1, \dots, 5$. Point out that the fixed points can be seen as horizontal lines, and the attractor can be seen as a horizontal line to which orbits starting in the basin of attraction converge. Also note that orbits are repelled from the non-attracting fixed point 1.

Now consider the function $f(x) = x^2 - 1$. Assign a different initial value to each pair of students. Use values such as $x_0 = -2, -1.5, -1, -.5, 0, .5, 1, 1.5, 2$. Have students find the first ten terms of the orbit and make a time-series graph.

Compare the results of different initial points. Students who started with $x_0 = 0, x_0 = -1$, or $x_0 = 1$ should immediately get the alternating sequence of numbers 0, 1, 0, 1,... Tell students that this is called a **two-cycle**. Those who started with $x_0 = -.5$ or $.5$ should get an orbit that tends toward that same two-cycle. Starting with $x_0 = -1.5$ or 1.5 results in behavior that is less clear. Using $x_0 = -2$ or 2 results in values that increase without bound.

What about other initial points? Tell students that it is time to use the computer to perform these lengthy calculations and create the graphs. Explain that the computer spreadsheet will calculate and graph orbits of the function $f(x) = x^2 + c$ for various values of c , with various initial points. So far, by hand they have done $c = 0$ and $c = -1$.

Have students open the spreadsheet file you have created and enter 0 for c in cell B1. Note that cell B4 contains the value of x_0 , which is currently 0. Also note that the entire orbit in column B consists of zeros, showing that 0 is a fixed point. Also examine the horizontal line in the time-series graph. Now have students change the value of x_0 in cell B4 and note how the orbit and graph change, giving the same results they got earlier when they did it by hand.

Now have them enter 0 for x_0 in cell B4, and -1 for c in cell B1. They should see the two-cycle obtained previously. Then they can change the values of x_0 to confirm earlier results. Have them explore different values of x_0 (keep $c = -1$ in cell B1) and see what conjectures they can come up with.

They should notice that for certain values of x_0 , the orbit converges to the two-cycle, but for others it diverges. In this case, the two-cycle is attracting, and the values of x_0 which lead to converging orbits are in its basin of attraction.

Have students search for the boundary of the basin of attraction; i.e., the values a and b such that orbits converge to the two-cycle when $a < x_0 < b$ and diverge when x_0 is outside that interval. Also have them search for fixed points. Eventually they will discover that a is approximately -1.618 and b is approximately 1.618. Also b is a fixed point, but a is not, and the other fixed point is close to -.618. Ask them why the basin of attraction is symmetric with respect to 0 [because x values are squared by the function].

Have students try to determine the fixed points exactly using algebra. If necessary, give them the hint that they must solve the equation $f(x) = x$. [Solving $x^2 - 1 = x$ using the quadratic formula leads to $x = (1 + \sqrt{5})/2$ or $x = (1 - \sqrt{5})/2$.] Have students type these solutions in as the initial value, and see what happens. Because the actual fixed points are irrational and cannot be represented exactly on the computer, and because they are both repelling, the orbit still eventually moves away from these initial points.

Now have students repeat the search for fixed points, attractors, and the basin of attraction for the function $f(x) = x^2 + 1$; i.e., with $c = 1$. They will find that all orbits diverge, and that there seems to be no fixed point. Have them solve $f(x) = x$ to find a fixed point. They should obtain a discriminant of -3, indicating there is no real fixed point.

Now have students repeat the process for the function $f(x) = x^2 - 2$; i.e., with $c = -2$. They should discover that -1 and 2 are repelling fixed points, and starting with $x_0 = 0$ or -2 leads to the fixed point 2, while starting with $x_0 = 1$ leads to the fixed point -1. Starting with a value of x_0 with absolute value greater than 2 leads to a diverging sequence.

However, most interestingly, starting with other values between -2 and 2 leads to apparently random sequences of numbers. Tell students this is called **chaos**, and has certain mathematical properties. One of them is sensitivity to initial conditions, more commonly referred to as the "butterfly effect," which can be demonstrated as follows. Have the two members of every pair of students type in slightly different values of x_0 ; e.g., .5 and .5001. They should notice that the orbits start out in a similar manner, but after a certain number of iterations eventually bear no resemblance to each other. Since chaos appears only when $-2 < x_0 < 2$, we can think of this as the basin of attraction for the attracting chaos.

Now have students keep x_0 set at 0, and experiment with different values of c . Have them record the different values they try, the resulting behavior (for example, a single attractor, an attracting cycle, or chaos), and the fixed points or cycles. You can assign certain ranges of values of c to different students or pairs of students. Also have students write down any observations they make. They should discover that all orbits diverge for $c > .25$ and $c < -2$ (except for those that start at fixed points), so the interesting results are when $-2 < c < .25$. They can also discover 4-cycles, 8-cycles, 3-cycles, 6-cycles, 5-cycles, and in fact, cycles of any period. Chaos can also be found for many values of c .

Synthesize the class's findings by having students contribute to a chart with the following headings. Have them asterisk the attractors.

c	Behavior	Fixed points	2-Cycles
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Some observations that students may make follow:

- As the value of c decreases, fixed points appear at $c = .25$.
- The sum of the fixed points is 1.
- For $-.75 < c < .25$, the smaller fixed point is attracting, and the larger is repelling, and the basin of attraction is $-p < x < p$ where p is the repelling fixed point.
- As c decreases, at $c = -.75$ both fixed points become repelling and an attracting 2-cycle appears.
- The numbers in the 2-cycle always add up to -1.

The first two observations can be explained algebraically. Have students find the fixed points for the general function $f(x) = x^2 + c$. They should use the quadratic formula to solve the equation $x^2 + c = x$, and obtain $x = (1 + \sqrt{1 - 4c})/2$ or $x = (1 - \sqrt{1 - 4c})/2$. Note that when $c > 1/4$ the discriminant is negative so there are no real fixed points, but when $c < 1/4$ the discriminant is positive so two real fixed points exist. (Calculus is needed to explain why one is attracting and the other is repelling.) Also, the sum of these two fixed points is $(1 + \sqrt{1 - 4c})/2 + (1 - \sqrt{1 - 4c})/2 = 2/2 = 1$.

Assessment:

There are two options. The less comprehensive one is to assign students a particular value of c and have them use the spreadsheet to investigate all possible behaviors of orbits of $f(x) = x^2 + c$. Students should find fixed points, determine whether they are attracting or repelling, and find the basin of attraction. They should explain their numerical findings algebraically if possible.

The second option is more comprehensive. Have students write a paper which describes their investigation and findings throughout this activity. They should explain the terms iteration, initial point, orbit, fixed point, cycle, chaos, and basin of attraction. They should explain the different kinds of behavior that can result from iteration, how it can be graphed, and how it depends on the values of c and the initial point. They should give specific examples. They should also describe the patterns and relationships they noticed. These findings should be explained by algebra when possible.

Either option can be scored according to a holistic rubric, evaluating the students' work in five areas:

- (1) Description of mathematical concepts involved
- (2) Explanation of procedures used
- (3) Description of findings
- (4) Use of notation and algebraic manipulation
- (5) Logical reasoning

Each of these areas can be scored on the following scale:

- 4 Correct and complete.
- 3 Almost correct and complete; some errors made or details omitted.
- 2 Shows general understanding, but notable gaps or errors are present.
- 1 Some work is correct, showing minimal understanding, but there is little or no chain of reasoning.
- 0 Wrong or meaningless; no evidence of understanding.

Extension/Follow Up:

Using more advanced algebra, students could analyze two-cycles analytically by solving the equation $f(f(x)) = x$. This leads to the fourth degree equation $(x^2 + c)^2 - x + c = 0$. Since fixed points (which are solutions to the equation $x^2 - x + c = 0$) also solve the fourth degree equation, long division can be used to find another quadratic equation which can be solved to find a formula for two-cycles.

This could be used to explain why a 2-cycle appears at $c = -3/4$, and why the sum of the values of the 2-cycle is -1 .

Another option uses calculus. The derivative of f can be used to explain why for $-3/4 < c < 1/4$, the greater fixed point is repelling while the smaller one is attracting. This uses the fact that when the derivative at a fixed point is less than 1 in absolute value, then the point is attracting, and when it is greater than 1 in absolute value, the point is repelling.

Yet a third option is to graph attracting fixed points, cycles, and chaos against the value of c . This results in what is often called the orbit diagram or bifurcation diagram. Then the period doubling phenomenon (i.e., whereby 2-cycles become 4-cycles then 8-cycles, etc.) can be studied.

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